

Problem Session 3

02/02/2018

(1) Problem 3.9, Jackson.

(2) Problem 3.12 (a), (b), Jackson.

(2)

(1) Vanishing the potential at $z=0$ and $z=L$ requires that:

$$Z(z) \propto \sin\left(\frac{n\pi}{L}z\right) \Rightarrow k_n^2 = -\frac{n^2\pi^2}{L^2}$$

For $k^2 < 0$, the potential involves modified Bessel functions I_m, K_m .

However, since K_m blows up at $\rho=0$, we have the following general

form for the potential:

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} I_m(k_n \rho) e^{im\phi} \sin\left(\frac{n\pi}{L}z\right)$$

The boundary condition at $\rho=a$ implies that:

$$\sum_{m,n} A_{mn} I_m(k_n a) e^{im\phi} \sin\left(\frac{n\pi}{L}z\right)$$

This is a double Fourier series over ϕ and z . Then:

$$A_{mn} = \frac{1}{\pi L I_m\left(\frac{n\pi}{L}a\right)} \int_0^L dz \int_0^{2\pi} d\phi V(\rho, z) e^{-im\phi} \sin\left(\frac{n\pi}{L}z\right)$$

Substituting this in the above expression for Φ gives the

requested series expansion.

(2) (a) First of all, due to azimuthal symmetry, there is no ϕ -dependence in the potential. Also, since the potential is due to a localized distribution, it must vanish at infinity. For $z > 0$, this requires that $Z(z) \propto e^{-kz}$ where $k > 0$.

As a result, the most general form of the potential is:

$$\Phi(s, z) = \int_0^{\infty} A(k) J_0(ks) e^{-kz} k dk$$

(b) For $z = 0$, we have:

$$\Phi(s, 0) = \int_0^{\infty} A(k) J_0(ks) k dk = \begin{cases} V & s < a \\ 0 & s > a \end{cases}$$

Using the Hankel transform relation gives:

$$A(k) = \int_0^{\infty} \Phi(s, 0) J_0(ks) s ds = \frac{V}{k} \int_0^a \underbrace{J_0(ks) ks ds}_{\frac{d}{d(ks)} (J_1(ks) ks)} = \frac{Va}{k} J_1(ka)$$

Therefore:

$$\Phi(s, z) = Va \int_0^{\infty} J_1(ka) J_0(ks) e^{-kz} dk$$

For P_{50} , we have $J_0(kr) \leq 1$. Hence;

$$\Phi(r, z) = \sqrt{a} \int_0^{\infty} J_1(ka) e^{-kz} dk$$

$\frac{1}{a} \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right)$ ← from integral tables

This results in;

$$\Phi(r, z) = \sqrt{a} \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right)$$